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Gyrokinetic Equations For Strong-Gradient Regions

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GYROKINETIC EQUATIONS FOR STRONG-GRADIENT REGIONS

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The gyrokinetic derivation of [A.M. Dimits, L.L. LoDestro, D.H.E. Dubin, Phys. Fluids B**4**, 274 (1992).] is extended to general equilibrium magnetic fields. The result is a practical set equations that is valid for large perturbation amplitudes [$q\psi / T = O(1)$, where $\psi = \phi - v_{\|}\delta A_{\|}/c$] but which is much simpler, easier to implement, and has more straightforward expressions for its conservation properties than the equation sets derived in the large-flow orderings. Here, ϕ and $\delta A_{\|}$ are the perturbed electrostatic and parallel magnetic potentials, $v_{\|}$ is the particle velocity, c is the speed of light, and T is the temperature. The derivation is based on the quantity $\varepsilon \equiv \left(\rho/\lambda_{\perp}\right) q\psi / T \ll 1$ as the small expansion parameter, where ρ is the gyroradius and $\lambda_{\perp}$ is the perpendicular wavelength. Physically, this means that the $E \times B$ velocity and the component of the parallel velocity perpendicular to the equilibrium magnetic field are small compared to the thermal velocity. For nonlinear fluctuations saturated at mixing-length levels (e.g., with $q\psi / T \sim \lambda_{\perp}/L$, where L is the equilibrium profile scale length), ε is of order ρ / L for all scales $\lambda_{\perp}$ ranging from ρ to L, even though $q\psi / T = O(1)$ for $\lambda_{\perp} \sim L$.

I. Introduction

Turbulence driven by instabilities with frequencies well below the ion cyclotron frequency and wavelengths of order or longer than the ion gyroradius are believed to be a key cause of anomalous transport that is observed in magnetic fusion plasmas. The presence of a strong ambient magnetic field in a plasma can, in many cases, be exploited to simplify the description of the plasma and the prediction of its behavior. In particular, if there is a dominant (both spatially and temporally) slowly varying magnetic field such that the gyro- (or "cyclotron" or "Larmor") orbital motion of a charged particle is much more rapid than the rate of change of the electromagnetic fields (and the rate of any "scattering" processes) seen by the particle, then the particle's magnetic moment is an adiabatic invariant. This fact was the basis for guiding-center theory[1] and, later, gyrokinetic theory.[2-8] In both cases, the temporal variation of the system is taken to be slow compared to the gyro frequency. In the guiding-center theory, the particle gyro-orbit size is required to be small compared to all spatial scales of any inhomogeneities in the system, while the gyrokinetic theory permits perturbations with scales comparable to the gyro orbit size.

Gyrokinetic-equation-based models have found wide use in the simulation of microturbulence and the resulting transport in magnetic fusion core plasmas.[6, 9-17] The gyrokinetic equations are valid under certain "gyrokinetic orderings," are faithful to the key kinetic (i.e., non-hydrodynamic) physics, but reduce the dimensionality of the relevant phase space by one relative to the raw "full-dynamics" Vlasov-Fokker-Planck equation. This reduction in dimensionality (e.g., from 6 to 5 for spatially 3-dimensional systems) can result in a large corresponding reduction in the number of degrees of freedom (grid cells, nodes, basis functions, or particles) needed to discretize the phase-space density ("distribution") function to a given level of accuracy. Accompanying this reduction is the removal of various high-frequency modes that are often not of central interest for the microturbulence or transport processes being simulated, but which may be numerically problematic.[6]

The success of gyrokinetic simulation of magnetic fusion energy (MFE) core plasmas has motivated interest in extending such models to the (outer) edge and scrapeoff-layer regions,[18,19] as well as to other situations that stress the existing orderings. The length-scale separation between the radial plasma scales and the gyroradius scales in the MFE edge region is much less than in the core. Thus, while we expect that a gyrokinetic ordering may still be satisfied in many edge and scrapeoff-layer situations, it will be less easily satisfied than in the core. Additional care therefore needs to be taken to ensure that the particular expression and use of the ordering results in a set of equations that are valid for the conditions and phenomena expected in the edge.

Some progress in this direction has been made by several authors.[20-24] In Ref. 20, the derivation of the gyrokinetic equations was extended to allow for electrostatic potential perturbations to be of order the temperature, and therefore relative perturbations of order 1 in the moments of the distribution function. The work of Refs. 21-24 allowed for large $E \times B$ flow velocities (of order thermal), as a separate static long-wavelength component. The seminal work in this direction by Artun and Tang[21] allowed for a large static axisymmetric equilibrium velocity $\boldsymbol{V}$ that satisfies $\boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{V} = 0$, and derived the appropriate gyrokinetic theory to first order (as is appropriate for core instability and turbulence studies) using an iterative approach.[4] Brizard[22] rederived and extended some of the results of Ref. 21 to second order using a Hamiltonian formalism that allowed for equilibrium flow components with $\boldsymbol{V}\cdot\boldsymbol{\nabla}\boldsymbol{V}\neq 0$. Further results and applications of Brizard's theory for electrostatic cases were derived by Hahm[23] and for electromagnetic cases by Hahm and Madsen.[24] Dimits[25] rederived the gyrokinetic equations using a one-step method that removed the static and two-scale restrictions of Refs. 21-24, and made all (not just the short-scale components) of the perturbations consistent with the evolving distribution function. This derivation showed that at second order, additional terms should be present in the Lagrangian under the assumptions and orderings of Refs. 21-24. These terms are cross terms involving the long-wavelength $E\times B$ flow and the gyrophase dependent part of the short-wavelength potential, and were absent because of the two-step nature of the derivations employed. Subsequently, Parra and Calvo[26] found that many such terms were needed in the second-order gyrokinetic theory, even in the standard "core-plasma" orderings.

However, the fully consistent large-flow-ordering gyrokinetic equations derived in Ref. 25 are complicated and provide several unresolved challenges for numerical implementation, even in the simplified slab electrostatic case addressed in Ref. 25. First, because of several new terms in the symplectic (non-canonical) part of the Lagrangian, the equations of motion are much more challenging to compute from the Lagrangian. Second, the expression of the conserved energy as a manifestly positive-definite quantity in terms of the gyrocenter variables is more challenging.

It is therefore of interest to investigate slightly less general orderings that may result in gyrokinetic equations that are more easily implemented. The ordering of Ref. 20 is such an ordering, and is expected to be adequate for edge and scrapeoff-layer conditions provided that the $E\times B$ drift velocities are much less than the thermal velocity. However, the derivation in Ref. 20 was for the simplified case of a slab magnetic field, although it allowed for electromagnetic perturbations. The present work extends this derivation to general (including toroidal) equilibrium magnetic fields.

The fundamental requirement for the magnetic moment of a charged particle in a magnetic field to be an adiabatic invariant can be written as

$$\varepsilon=\frac{\omega}{\Omega}\ll 1\,,\quad (1)$$

where ω is the rate of change of the electromagnetic fields seen by the particle and $\Omega=qB/Mc$ is the cyclotron frequency. Here, q and M are the particle charge and mass, B is the magnetic field strength, and c is the speed of light. The magnetic moment can be written to lowest order as $\mu=Mv_\perp^2/2B$, where $v_\perp$ is the magnitude of the velocity perpendicular to the magnetic field.

Most nonlinear gyrokinetic theories[4-8] have as requirements that

$$\frac{q\delta\phi}{T}\sim\frac{\delta f}{F_{\rm eq}}\sim k_{\|}\rho_t\sim\varepsilon\ll 1\,,\quad (2)$$

or an electromagnetic generalization thereof. Here, $\delta\phi=\delta\phi(\boldsymbol{x},t)$ is the perturbed electrostatic potential, T is the temperature, δf and $F_{\rm eq}$ are the perturbed and equilibrium phase-space distribution functions, $k_\parallel$ is the characteristic parallel wavenumber, $\rho_t = v_t/\Omega$ is the thermal gyroradius, $v_t = \sqrt{T/M}$ is the thermal (kinetic) particle speed, and ε is the formal expansion parameter ("gyrokinetic smallness parameter"). The first condition of Eq. (2) is somewhat counterintuitive, as it suggests that the theory would break down if a large constant were added to $\delta\phi$. A partial resolution of this apparent paradox lies in that these theories also formally order $k_\perp\rho_t \sim 1$, where $k_\perp$ is the characteristic perpendicular wavenumber. Equation (2) also formally rules out the application of the resulting gyrokinetic system to situations where large perturbations are present, for example in the outer edge region of magnetically confined plasmas, even if the frequency ordering for adiabatic invariance of the magnetic moment is satisfied.

Dimits et. al.[20] extended the canonical Hamiltonian gyrokinetic theory[5,8] to allow for $q\delta\psi/T = O(1)$, where $\delta\psi = \delta\phi - v_\parallel \delta A_\parallel/c$, $\delta A_\parallel$ is the perturbed parallel magnetic potential, $v_\parallel$ is the parallel particle velocity, and c is the speed of light. Ref. 20 is based on the new small parameter

$$\varepsilon_{\mathrm{V}} \sim \frac{V_\psi}{v_{\mathrm{th}}} \simeq k_\perp \rho_t \frac{q\delta\psi}{T} \ll 1 . \qquad (3)$$

Here V_ψ is a characteristic drift velocity associated with $\delta\psi$, and includes the $E\times B$ drift velocity associated with $\delta\phi$ as well as the component of the parallel velocity perpendicular the equilibrium magnetic field. Under this ordering, $\delta\psi$ can have large long-wavelength components and small short-wavelength components, as well as components of intermediate sizes at intermediate scales. In the present paper, we extend this derivation to general (including toroidal) equilibrium magnetic fields using noncanonical Hamiltonian methods.

Parra and Catto[27] derived the gyrokinetic equations for the case of a general equilibrium magnetic field and electrostatic under this ordering using an iterative method. The present work can be viewed as an extension of the derivation of Ref. 20 to include spatial variation in the equilibrium magnetic field, and of Ref. 27 to allow for electromagnetic perturbations. An advantage of our derivation compared to those that use the iterative approach is that through the use of a Hamiltonian method, our derivation easily provides conservation laws through Noether's theorem and a variational formulation.

A clear exposition of the basic noncanonical Hamiltonian perturbation method, along with the application to the derivation of the drift-kinetic equations for a particle in an inhomogeneous magnetic field was given by R. Littlejohn.[28] This method was applied to the derivation of the gyrokinetic particle (characteristic), Vlasov, and Poisson equations for a plasma in a toroidal magnetic field by Hahm. [7] Subsequently, generalizations were made to a variety of situations, and this body of work has been reviewed by Brizard and Hahm.[8]

We anticipate that our equations will be useful as a basis for simulation models for strong gradient regions in magnetized plasmas, e.g., edge and screapeoff layer regions as well as internal transport barriers where the separation between the profile and gyroradius scales is relatively modest, and the significant fluctuations have wavelengths (of the order of 10-100 gyroradii) that may be comparable to the scale of profile variations.

II. Gyrokinetic equations in the drift ordering

We consider a plasma in an inhomogeneous time dependent magnetic field $\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{B}_0(\boldsymbol{x}) + \delta\boldsymbol{B}(\boldsymbol{x},t)$, where $\boldsymbol{B}_0(\boldsymbol{x}) = B_0(\boldsymbol{x})\hat{\boldsymbol{b}}_0(\boldsymbol{x})$ is an equilibrium magnetic field, and we define $B_0(\boldsymbol{x}) = |\boldsymbol{B}_0(\boldsymbol{x})|$, so that $\hat{\boldsymbol{b}}_0(\boldsymbol{x})$ is the unit vector in the direction of $\boldsymbol{B}_0$. We allow for electromagnetic perturbations consisting of a perturbed electrostatic potential $\delta\phi = \delta\phi(\boldsymbol{x},t)$ and a perturbed magnetic field in a low-β ordering, where β is the ratio of plasma to magnetic pressure, $\delta\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{\nabla}\times\left[\delta A_{\|}(\boldsymbol{x},t)\hat{\boldsymbol{b}}_0(\boldsymbol{x})\right]$. The generalized potential $\delta\psi = \delta\phi - v_{\|}\delta A_{\|}$ is taken to satisfy $\hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\delta\psi\big/(\Omega_0 v_{\rm th}) \ll 1$, where $\Omega_0 = qB_0 / Mc$ is the gyrofrequency, q is the charge, M is the mass, and c is the speed of light. The electrostatic part of this ordering relation specifies that the ratio of the $E\times B$ velocity $\boldsymbol{V}_{E\times B} = c\hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\delta\phi\big/B_0$ to the thermal speed is small, and the magnetic potential part specifies that the perpendicular velocity associated with magnetic flutter is small compared to the thermal speed or, equivalently, that the angle between the perturbed and unperturbed field lines is small.

Additionally a strong-gradient-region ordering will be used for the perturbations and magnetic equilibrium. Under this ordering it is assumed that the spatial scale L_p over which the perturbed quantities can undergo an order-1 change is much shorter than the scale over which the magnetic equilibrium changes, which for example in a tokamak is typically the major radius R. Thus

$$L_p\big/R \ll 1.$$

The basic gyrokinetic ordering Eq.(5), applied to fluctuations of gyroradius scale implies the condition

$$\rho\big/L_p \ll 1.$$

From typical mixing-length saturation rules, L_p can be associated with the profile scales associated with the driving gradients of the instabilities or associated turbulence under study. We will therefore use the following combination of the gyrokinetic and strong-gradient-region orderings

$$\varepsilon \sim \rho\big/L_p \sim L_p\big/R \ll 1. \qquad (4)$$

In the absence of the strong-gradient ordering, many unfamiliar terms arise at second order in the theory. These terms involve beating of the gyrophase dependent terms involving equilibrium magnetic field variation and the perturbations. This difficulty arises even in the "standard" gyrokinetic ordering [Calvo, 2010]. Instead of working with the combined orderings of Eq. (4), one could avoid these terms by working only to first order in the magnetic equilibrium dependences, irrespective of the order to which gyrokinetic expansion is taken on the perturbations. However, the use of the orderings in Eq. (4) is more systematic and transparent with respect to the validity of the resulting theory. Consistent with Eq.(4), and having the equilibrium magnetic field at zero order, we take

$$qA_0 / (Mcv_{\mathrm{th}}) = O(\varepsilon^{-2}).$$

Because the perturbed ($E \times B$ and magnetic-flutter) velocities are small, they will not have a zero-order contribution to the symplectic (noncanonical) terms in the particle Lagrangian. The standard two-step procedure for deriving the Lagrangian can therefore be used. This begins with the unperturbed guiding-center phase-space variables

$$Z = (\boldsymbol{R}, U_{\|}, \mu, \theta),$$

where $\boldsymbol{R}$ is the guiding center position, $U_{\|}$ is the parallel velocity μ is the magnetic moment, and θ is the gyrophase angle. These evolve according to the Euler-Lagrange equations with the standard first order guiding-center phase-space Lagrangian for the motion of a charged particle in a static inhomogeneous magnetic field [Littlejohn]:

$$L(Z, \dot{Z}, t) = \left[\boldsymbol{A}_0(\boldsymbol{R}) + U_{\|}\hat{\boldsymbol{b}}_0(\boldsymbol{R})\right] \cdot \dot{\boldsymbol{R}} - \mu\dot{\theta} - \left[\frac{U_{\|}^2}{2} + \mu\Omega(\boldsymbol{R})\right]. \qquad (5)$$

We normalize energies and the Lagrangian to the thermal energy, ($L \sim T$), velocities (e.g., $U_{\|}$) to the mean thermal velocity v_{th}, momenta to Mv_{th}, magnetic potentials ($\boldsymbol{A}$, $\delta\boldsymbol{A}$) to the quantity $Mv_{\mathrm{th}}c/q$. Electrostatic potentials (ϕ) will be normalized to T/q. It can easily be checked that the Euler-Lagrange equations applied to Eq. (5) give the standard guiding-center equations of motion with parallel streaming, and the $\boldsymbol{\nabla}B$ and curvature drifts.

To the Lagrangian of Eq. (5), we add the perturbed Lagrangian

$$\delta L = \delta A_{\|}(\boldsymbol{x}, t)\hat{\boldsymbol{b}}_0(\boldsymbol{x}) \cdot d\boldsymbol{x} - \delta\phi(\boldsymbol{x}, t)\, dt. \qquad (6)$$

There are two standard choices for the perturbed guiding-center phase-space variables. In the "symplectic representation," $U_{\|}$ is used as the parallel momentum variable, while in the "canonical representation," a perturbed canonical momentum $p_{\|} = U_{\|} + \delta A_{\|}$ is used. Thus, in the canonical representation the perturbed guiding-center coordinates and Lagrangian are $Z = (\boldsymbol{R}, p_{\|}, \mu, \theta)$ and

$$L(Z, \dot{Z}, t) = \left[\boldsymbol{A}_0(\boldsymbol{R}) + p_{\|}\hat{\boldsymbol{b}}_0(\boldsymbol{R})\right] \cdot \dot{\boldsymbol{R}} - \mu\dot{\theta} - \left\{\frac{\left[p_{\|} - \delta A_{\|}(\boldsymbol{R} + \boldsymbol{\rho}, t)\right]^2}{2} + \mu\Omega(\boldsymbol{R}) + \delta\phi(\boldsymbol{R} + \boldsymbol{\rho}, t)\right\},$$

while in the symplectic representation they are $Z = (\boldsymbol{R}, U_{\|}, \mu, \theta)$ and

$$L(Z, \dot{Z}, t) = \left\{\boldsymbol{A}_0(\boldsymbol{R}) + \left[U_{\|} + \delta A_{\|}(\boldsymbol{R} + \boldsymbol{\rho}, t)\right]\hat{\boldsymbol{b}}_0(\boldsymbol{R})\right\} \cdot \dot{\boldsymbol{R}} - \mu\dot{\theta} - \left[\frac{U_{\|}^2}{2} + \mu\Omega(\boldsymbol{R}) + \delta\phi(\boldsymbol{R} + \boldsymbol{\rho}, t)\right].$$

We now proceed in the canonical representation, as this is somewhat simpler than the symplectic representation. First, separate $\delta\phi$ and $\delta A_{\|}$ as

$$\delta\phi(\boldsymbol{R} + \boldsymbol{\rho}, t) = \delta\bar{\phi}(\boldsymbol{R}, t) + \delta\tilde{\phi}(\boldsymbol{R}, \mu, \theta, t)$$

and

$$\delta A_{\parallel}(\boldsymbol{R}+\boldsymbol{\rho},t)=\delta\overline{A}_{\parallel}(\boldsymbol{R},t)+\delta\tilde{A}_{\parallel}(\boldsymbol{R},\mu,\theta,t)$$

where $\delta\overline{\phi}(\boldsymbol{R},\mu,t)$ and $\delta\overline{A}_{\parallel}(\boldsymbol{R},\mu,t)$ are gyrophase-independent approximations to $\delta\phi$ and $\delta A_{\parallel}$. A convenient choice for these, and one which will simplify the subsequent derivations is

$$\begin{aligned}\delta\psi(\boldsymbol{R},t)&=\langle\delta\psi\rangle\\&=\frac{1}{2\pi}\oint d\theta\delta\psi(\boldsymbol{R}+\rho\hat{\boldsymbol{\rho}}(\theta),t)\end{aligned}$$

where $\rho=\sqrt{\dfrac{2\mu}{\Omega(\boldsymbol{R})}}$, and $\hat{\boldsymbol{\rho}}(\theta)$ is a unit vector perpendicular to $\hat{\boldsymbol{b}}_0(\boldsymbol{R})$ and which subtends an angle θ with respect to a fixed plane containing $\hat{\boldsymbol{b}}_0(\boldsymbol{R})$. Then

$$\delta\tilde{\phi}\equiv\phi(\boldsymbol{R}+\boldsymbol{\rho},t)-\overline{\phi}(\boldsymbol{R},\mu,t)\approx\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\overline{\phi}=O\left(\varepsilon_{\mathrm{V}}=V_{E\times B}/v_{\mathrm{th}}\right),\qquad(7)$$

where ε_{V} is the small ordering parameter, as given in the ordering relation Eq. (1). Similarly,

$$\delta\tilde{A}_{\parallel}\equiv\delta A_{\parallel}(\boldsymbol{R}+\boldsymbol{\rho},t)-\delta\overline{A}_{\parallel}(\boldsymbol{R},\mu,t)\approx\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\delta\overline{A}_{\parallel}=O\left(\varepsilon_{\mathrm{V}}=\delta B_{\perp}/B_0\right).\qquad(8)$$

After the transformation to the perturbed guiding-center variables and the above separation of $\delta\tilde{\phi}$ and $\delta\tilde{A}_{\parallel}$, the resulting Lagrangian is

$$\begin{aligned}L&=\langle\boldsymbol{A}_0\rangle\cdot\dot{\boldsymbol{R}} &&\ldots\ldots O\left(\varepsilon^{-2}\right)\\&+p_{\parallel}\hat{\boldsymbol{b}}_0\cdot\dot{\boldsymbol{R}}-\mu\dot{\theta}-\left[\frac{1}{2}\left(p_{\parallel}-\delta\overline{A}_{\parallel}\right)^2+\mu\Omega+\delta\overline{\phi}\right] &&\ldots\ldots O\left(\varepsilon^{0}\right)\\&+\left[\delta\tilde{\phi}-\left(p_{\parallel}-\delta\overline{A}_{\parallel}\right)\delta\tilde{A}_{\parallel}\right] &&\ldots\ldots O\left(\varepsilon^{1}\right)\\&-\frac{1}{2}\left(\delta\tilde{A}_{\parallel}\right)^2\cdot\dot{\boldsymbol{R}}, &&\ldots\ldots O\left(\varepsilon^{2}\right)\end{aligned}\qquad(9)$$

where the formal order of each term in our expansion parameter is shown, and we have renamed ε_{V} as ε. The terms in Eq.(9) are separated into terms formally of order ε^{-2} through ε^{2}. The key result of the separation in Eqs. (7) and (8) is that the only gyrophase dependent terms in Eq. (9) are at orders ε^{1} and higher.

A Lie-transform perturbative treatment is applied to transform the phase-space coordinates $Z\to\overline{Z}=\left(\overline{\boldsymbol{R}},\overline{U}_{\parallel},\overline{\mu},\overline{\theta}\right)$, to eliminate the gyrophase dependence in the Lagrangian of Eq. (9) using $\delta_1\tilde{\phi}\sim\varepsilon\ll 1$. The derivation is similar to the standard noncanonical Hamiltonian Lie transform perturbation theory,[8,27] but also has some differences. The coordinate transformation is represented as an operator $T(\varepsilon)$ as the action of which on the coordinates, distribution function, and Poincare-Cartan one-form γ defined by $\gamma = L\,dt$, where L is the Lagrangian.

$$
\begin{aligned}
& Z \to \bar{Z} = \left(\bar{\boldsymbol{R}}, \bar{U}_{\parallel}, \bar{\mu}, \bar{\theta}\right) = T(\varepsilon) Z, \\
& f(Z) = \bar{f}\left(\bar{Z}\right) = T\,\bar{f}(Z), \\
& \Gamma \equiv \bar{\gamma} = T^{-1}\gamma + dS.
\end{aligned}
$$

$T(\varepsilon)$ is further represented as a product of operators, each of which is formally an exponential of Lie derivative operators that acts only at successively higher orders

$$
\begin{aligned}
& T = \ldots\ldots T_3 T_2 T_1, \\
& T_n(\varepsilon) = \exp\left(\varepsilon^n L_n\right).
\end{aligned}
$$

Here the L_n 's are Lie derivatives (not to be confused with the Lagrangian), the action of which on a scalar and one form are

$$
\begin{aligned}
& L_n \Lambda = g_n^{\beta} \frac{\partial \Lambda}{\partial Z^{\beta}}, \\
& \left(L_n \gamma\right)_{\alpha} = g_n^{\beta} \left(\frac{\partial \gamma_{\alpha}}{\partial Z^{\beta}} - \frac{\partial \gamma_{\beta}}{\partial Z^{\alpha}}\right).
\end{aligned} \tag{10}
$$

Operators involving spatial derivatives are assigned different orders depending on the quantity they operate on.[20] Spatial derivatives acting on a quantity at a given order in the Lagrangian of Eq. (9) may demote that quantity zero, one or two orders. We therefore, write

$$
L_n \gamma = \left(L_n \gamma\right)_a + \varepsilon\left(L_n \gamma\right)_b + \varepsilon^2\left(L_n \gamma\right)_c
$$

The main results needed from the perturbation theory to obtain the Lagrangian up to second order are

$$
\begin{aligned}
& \Gamma_{-2,-1,0} = \gamma_{-2,-1,0} + dS_{-2,-1,0}, \\
& \Gamma_1 = \gamma_1 - \left(L_1 \gamma_0\right)_a - \left(L_1 \gamma_{-2}\right)_c + dS_1, \\
& \Gamma_2 = \left\langle \gamma_2 \right\rangle - \frac{1}{2}\left\langle \left(L_1 \gamma_1\right)_a \right\rangle
\end{aligned} \tag{11}
$$

The key steps in the evaluation of these are presented in the Appendix.

The resulting phase-space Lagrangian for the gyrocenter motion, in the canonical representation for the magnetic perturbations, up to second order is

$$
\begin{aligned}
L = \left[\langle \boldsymbol{A}_0\rangle + p_\parallel \hat{\boldsymbol{b}}_0\right]\cdot\dot{\boldsymbol{R}} - \mu\dot{\theta} - \Bigg[\frac{1}{2}\left\langle\left(p_\parallel - \delta A_\parallel\right)^2\right\rangle + \mu\Omega + \langle\phi\rangle \\
+ \frac{1}{2\Omega_0}\left\langle \boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega_0\right)\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\tilde{\psi}\right\rangle - \frac{1}{2\Omega_0}\frac{\partial}{\partial\mu}\left\langle\tilde{\psi}^2\right\rangle\Bigg],
\end{aligned}
\tag{12}
$$

where

$$
\tilde{\psi} = \tilde{\phi} - \left(p_\parallel - \delta\overline{A}_\parallel\right)\delta\tilde{A}_\parallel,
$$

$$
\Psi = \int^{\theta}\tilde{\psi}\,d\theta,
$$

$$
\tilde{\Psi} = \Psi - \langle\Psi\rangle,
$$

and

$$
\langle\Psi\rangle = \frac{1}{2\pi}\oint\tilde{\Psi}\,d\theta.
$$

This is essentially the same result as has been obtained in the standard ordering,[8] but now we have provided a justification for it in the combination of the edge ordering of Eq. (4) for the equilibrium and the ordering of Eq. (3) for the perturbations, which is relevant to the edge of MFE devices.

From the Lagrangian of Eq. (12), we can verify some familiar results. Up to first-order the equations of motion that result from the $\boldsymbol{R}$ Euler-Lagrange equations give contain the $\boldsymbol{\nabla}B$, curvature, and $E\times B$ drifts. The parts of the Lagrangian of Eq. (12) up to first order are

$$
L_{-2,0,1} = \left[\boldsymbol{A}_{gc} + p_\parallel\hat{\boldsymbol{b}}_0\right]\cdot\dot{\boldsymbol{R}} - \mu\dot{\theta} - \left[\frac{1}{2}\left(p_\parallel - \left\langle\delta A_\parallel\right\rangle\right)^2 + \mu\Omega_0 + \langle\phi\rangle\right],
$$

and the $\boldsymbol{R}$ Euler-Lagrange equations give

$$
\dot{\boldsymbol{R}} = \left(p_\parallel - \delta A_\parallel\right)\hat{\boldsymbol{b}}_0 + \frac{1}{\Omega_0}\hat{\boldsymbol{b}}_0\times\left[\boldsymbol{\nabla}\langle\phi\rangle + \mu\,\boldsymbol{\nabla}\Omega_0 + \left(p_\parallel - \left\langle\delta A_\parallel\right\rangle\right)^2\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}_0\right]. \tag{13}
$$

The Vlasov equation for the evolution of the gyrocenter distribution function, neglecting collisions, can be obtained in the standard way using the fact that the absence of dependence of Lagrangian on θ decouples the gyrophase dependent and independent parts of the Vlasov equation[5]

$$
\frac{\partial F_i}{\partial t} + \dot{\boldsymbol{R}}\cdot\boldsymbol{\nabla}_R F_i + \dot{U}_\parallel\frac{\partial F_i}{\partial U_\parallel} = 0. \tag{14}
$$

In a self-consistent model, the gyrokinetic species described by the above equations will contribute to the field equations through its density n_i and current $\boldsymbol{J}_i$. These are obtained from

F_i via the "pullback transformation"[5] and integration over velocity space. The results are for the density n_i

$$n_i = \int \Omega dZ \ \delta(\boldsymbol{R} + \rho - \boldsymbol{x})\left[F_i(\boldsymbol{R}, \mu) + \frac{1}{\Omega}\delta\tilde{\psi}\frac{\partial F_i}{\partial \mu} + \frac{1}{\Omega}\boldsymbol{\nabla}_{R\perp}\left(\frac{\tilde{\Psi}}{\Omega}\right)\cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp} F_i\right] \tag{15}$$

and

$$\delta J_{\|i} \approx Z_i e \int d\Lambda \delta(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x})\, p_{\|}\left(F_i + \delta\tilde{\psi}\frac{\partial F_i}{\partial \mu} + \boldsymbol{\nabla}_{R\perp}\delta\tilde{\Psi}\cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp} F_i\right). \tag{16}$$

Inserting Eq. (15) and (16) respectively into the Poisson equation and Ampere's law, for the case of a single gyrokinetic ion species, and neutralizing electrons gives

$$\begin{aligned}\nabla^2 \phi &= 4\pi e n_{\mathrm{e}} \\ &\quad - 4\pi Z e \int \Omega dZ \ \delta(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x})\left[F_i + \frac{1}{\Omega}\delta\tilde{\psi}\frac{\partial F_i}{\partial \mu} + \frac{1}{\Omega}\boldsymbol{\nabla}_{R\perp}\left(\frac{\tilde{\Psi}}{\Omega}\right)\cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp} F_i\right]\end{aligned} \tag{17}$$

and

$$\begin{aligned}\left(\nabla^2 / c^2\right) A_{1\|} &= \left(4\pi e / m_e\right)\int dV F_e p_{\|} - \left(4\pi Z e / m_i\right)\times \\ &\int \Omega dZ \delta(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x})\, p_{\|}\left(F_i + \frac{1}{\Omega}\delta\tilde{\psi}\frac{\partial F_i}{\partial \mu} + \frac{1}{\Omega}\boldsymbol{\nabla}_{R\perp}\left(\frac{\tilde{\Psi}}{\Omega}\right)\cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp} F_i\right).\end{aligned} \tag{18}$$

Equations (12) and the associated Euler-Lagrange equations, along with Eqs. (14), (17), and (18) constitute a closed gyrokinetic Vlasov-Maxwell system valid for strong-gradient regions in a magnetized plasma, under the small velocity ordering of Eq. (3) and the strong-gradient ordering of Eq. (4).

III. Summary

We have derived the (low-β) toroidal electromagnetic gyrokinetic equations in a new more general ordering which allows for large perturbation amplitudes and is appropriate for MFE edge plasma conditions. While the resulting equations are similar to already published, the present work is valuable because it provides a theoretical basis for the application of these equations in strong-gradient regions in magnetized plasmas, for example in the edge region and internal transport barriers in a tokamak. In particular, our ordering and derivation show that the equations are valid in such regions, while previous Hamiltonian derivations used an ordering that precluded large perturbations.

Also, the use of a strong-gradient ordering of the plasma profiles relative to the magnetic-field inhomogeneities results in equations valid to second order without the many finite-gyroradius and beating terms that must be kept in the more common "core-plasma" orderings [calvo]. Thus, the simplicity of the more familiar second-order equations is maintained. Because the equations are valid to second order and come from a Hamiltonian formalism, exact energy and momentum conservation relations can easily be obtained from Noether's theorem variational formulations

[Sugama]. This is an advantage over iterative derivations, for which the conservation relations that result are typically approximate rather than exact.

We anticipate that our equations will be useful as a basis for simulation models edge and screapeoff layer regions as well as internal transport barriers in tokamaks, where the separation between the profile and gyroradius scales is relatively modest, and the significant fluctuations have wavelengths (of the order of 10-100 gyroradii) that may be comparable to the scale of profile variations.

Appendix

Here, we show the key steps in the calculations leading from Eqs. (9) and (11) to Eq. (12).

For orders ε^{-2}, ε^{-1}, and ε^{0} in Eq. (11), we can choose

$$\Gamma_{-2,-1,0} = \gamma_{-2,-1,0},$$
$$dS_{-2,-1,0} = 0.$$

At first order, we have

$$0 = \langle \gamma_1 \rangle = \Gamma_1 = \gamma_1 - \left(L_1\gamma_0\right)_a - \left(L_1\gamma_{-2}\right)_c + dS_1,$$

which yields

$$\begin{aligned}
\boldsymbol{g}_1^{\perp} &= -\frac{1}{\Omega_0}\hat{\boldsymbol{b}}_0 \times \boldsymbol{\nabla} S_1, \\
g_1^{\parallel} &= -\frac{\partial S_1}{\partial p_{\parallel}}, \\
g_1^{p_{\parallel}} &= \nabla_{\parallel} S_1, \\
g_1^{\mu} &= -\frac{\partial S_1}{\partial \theta}, \\
g_1^{\theta} &= \frac{\partial S_1}{\partial \mu}, \\
0 &= \frac{\partial S_1}{\partial t} + \Omega g_1^{\mu} + \left(p_{\parallel} - \delta\bar{A}_{\parallel}\right) g_1^{p_{\parallel}} - \delta\tilde{\psi},
\end{aligned}$$

where

$$\delta\tilde{\psi} \equiv \delta\tilde{\phi} - \left(p_{\parallel} - \delta\bar{A}_{\parallel}\right)\delta\tilde{A}_{\parallel}.$$

these, in turn, result in

$$\frac{\partial S_1}{\partial t}+\left(p_{\|}-\delta\overline{A}_{\|}\right)\nabla_{\|}S_1-\Omega\frac{\partial S_1}{\partial\theta}=\delta\tilde{\psi}. \qquad (19)$$

In the solution of Eq.(19) for the first-order gauge function S_1 we use $\partial S_1/\partial t\sim\varepsilon\Omega\partial_\theta S_1$. S_1 is needed to get the first-order generating functions and the expression for the second-order parts of the Lagrangian.

$$\begin{aligned}
&\left(\frac{dS_1}{dt}\right)_{\text{slow}}-\Omega\frac{\partial S_1}{\partial\theta}=\delta\tilde{\psi},\\
&S_1\approx-\tilde{\Psi}/\Omega,\\
&\tilde{\Psi}\equiv\Psi_i-\overline{\Psi}_i,\\
&\Psi_i=\int_{\theta_0}^{\theta}d\theta\,\delta\tilde{\psi},\\
&\delta\tilde{\psi}\equiv\delta\tilde{\phi}-\left(p_{\|}-\delta\overline{A}_{\|}\right)\delta\tilde{A}_{\|}
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{g}_1^{\perp}=\frac{1}{\Omega}\hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega\right),\\
&g_1^{\|}=-\Delta\tilde{A}_{\|}/\Omega,\\
&g_1^{p_{\|}}=-\nabla_{\|}\left(\tilde{\Psi}/\Omega\right),\\
&g_1^{\mu}=\left(\delta\tilde{\psi}/\Omega\right),\\
&g_1^{\theta}=-\frac{1}{\Omega}\frac{\partial\delta\tilde{\Psi}}{\partial\mu},\\
&\tilde{\Psi}\equiv\tilde{\Phi}-\left(p_{\|}-\delta\overline{A}_{\|}\right)\Delta\tilde{A}_{\|},\\
&\tilde{\Phi}\equiv\Phi_i-\overline{\Phi}_i,\\
&\Phi_i=\int_{\theta_0}^{\theta}d\theta\,\delta\tilde{\phi},\\
&\Delta\tilde{A}_{\|}=\Delta A_{\|i}-\Delta\overline{A}_{\|i},\\
&\Delta A_{\|i}=\int_{\theta_0}^{\theta}d\theta\,\delta\tilde{A}_{\|}.
\end{aligned}$$

The resulting Poincare-Cartan one-form (phase-space Lagrangian) for the gyrocenter motion, in the canonical representation for the magnetic perturbations, up to second order is

$$
\begin{aligned}
\Gamma &= \langle \boldsymbol{A}_0 \rangle \cdot d\boldsymbol{R} \\
&+ p_{\|}\hat{\boldsymbol{b}}_0 \cdot d\boldsymbol{R} - \mu\, d\theta - \left[\frac{1}{2}\left\langle \left(p_{\|} - \delta A_{\|}\right)^2 \right\rangle + \mu\Omega + \delta\langle\phi\rangle\right] dt \qquad (20)\\
&+ \frac{1}{2}\left[\frac{1}{\Omega}\frac{\partial\left\langle\tilde{\psi}^2\right\rangle}{\partial\mu} + \frac{1}{\Omega}\left\langle \hat{\boldsymbol{b}}_0 \times \boldsymbol{\nabla}\left(\frac{\Psi}{\Omega}\right)\cdot\boldsymbol{\nabla}\tilde{\psi}\right\rangle + \nabla_{\|}\left(\frac{1}{\Omega}\left\langle\delta\tilde{A}_{\|}\Psi\right\rangle\right)\right] dt
\end{aligned}
$$

The last ($\nabla_{\|}$) term in the square brackets in Eq. (20) can be neglected because it is third order in ε.

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